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Uniqueness of Stationary Distributions in Random Access Poisson Networks

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***Abstract*—This paper presents sufficient conditions for the existence and uniqueness of the limiting stationary distribution in random access Poisson networks with packet queueing and under Rayleigh fading and a general path loss model. This system model is traditionally used in the literature assuming uniqueness of the stationary distribution. We demonstrate here through a counterexample that this assumption might not always hold. From the sufficient conditions, an interesting and perhaps counterintuitive result emerged, that is, the arrival rate of packets per node greater than $1/\mathrm{e}$ guarantees uniqueness of the limiting distribution. When that is the case, then setting the medium access probability to 1 minimizes the proportion of unstable nodes.**

***Index Terms*—Wireless networks, stochastic geometry, queueing theory, fixed point theory, slotted ALOHA.**

I. Introduction

Several papers that use stochastic geometry and queueing theory to model large-scale wireless networks have recently been reported in the literature [1]. These works capture the spatio-temporal behavior of the network through the introduction of static elements in the mathematical model. The objective of this letter is to show that these models may not have unique limiting (stationary) distributions. This is true even for simple "well-behaved" models like the ones presented in [1]. Hence, we expect that this will also be true for more realistic theoretical models as well as for real-world deployments. Despite its importance, the uniqueness of the distribution is usually assumed without further considerations. To the best of our knowledge, this letter is the first contribution in the literature to present the non-uniqueness problem of stationary distributions in Poisson networks, which is mainly motivated by the works [2], [3].

In [2], a performance analysis of a high-mobility network with N classes of users is presented. The paper results are analytically tractable and can adapt to different network scenarios using a large number of classes. However, this framework is not enough for our purposes because it works under stability conditions for all users, and the non-uniqueness of a limiting distribution emerges when there is a portion of unstable queues in the network, caused, for example, by static elements.

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In [3], the authors tackle the difficult task of taking into account spatial correlation, which causes a portion of nodes to be unstable. For that, they consider the active link probability of receivers in the network conditioned to the distance to a typical receiver. They use mean-field approximation and perform queueing analysis in the steady state. Although desirable, the same approach is unsuitable for our study because it leads to nonanalytically tractable expressions, and thus, it makes some mathematical derivations prohibitive.

In this paper, different from the existing literature that usually (implicitly) assumes the uniqueness of a valid stationary distribution, we do prove its existence and further establish sufficient conditions for its uniqueness. When these conditions are not enough to guarantee uniqueness (since they are only sufficient), we provide a simple method to verify uniqueness. We also present the conditions for which increasing the transmit probability decreases the proportion of unstable queues, thus improving the performance of the network. Then, we show the existence of a scenario that has two very different limiting stationary distributions, which are validated through simulations.

The rest of the paper is organized as follows. Section II presents the system model, Section III shows the theoretical results of the paper, Section IV provides a counterexample to uniqueness of stationary distributions, and Section V concludes the paper.

II. System Model

We consider a Poisson bipolar network under slotted ALOHA protocol, where transmitter locations are distributed according to a homogeneous Poisson point process (PPP) $\Phi \subset \mathbb{R}^2$ of a spatial density $\lambda > 0$. Each transmitter is located at $X_i \in \Phi$, $i \in \mathbb{N}$, and has a dedicated receiver Y_i at a fixed distance $R_i = ||Y_i - X_i||$ and a uniformly distributed random direction. Time is slotted and denoted by $t \in \mathbb{N}$. The distances $\{R_i\}_{i\in\mathbb{N}}$ are defined at the first time slot independently and according to some proper distribution with a cumulative distribution function (CDF) $F_R : \mathbb{R}_+ \longrightarrow [0, 1]$. Thus, the receiver locations $\{Y_i\}_{i\in\mathbb{N}} = \hat{\Phi} \subset \mathbb{R}^2$ also form a homogeneous Poisson point process with a spatial density λ according to the Displacement Theorem [4, Th. 1.3.9.].

We assume unit transmit power for all users, signals subject to small-scale Rayleigh fading independent and identically distributed (iid) across space and time, and a general omnidirectional monotone decreasing path loss model $\ell : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that satisfies $\int_{\mathbb{R}_+} x\, \ell(x)\, \mathrm{d}x < \infty$ to guarantee that the interference is not infinite *almost surely*.

Packets arrive independently at each transmitter queue according to a Bernoulli process of parameter $a \in [0, 1]$. The queue service discipline is general, i.e., it does not matter in which order the queued packets are transmitted, because this study deals with the distribution of queued packets and not queueing delay. If the transmitter has a nonempty queue, then it tries to transmit with an access probability $p \in [0,1]$. We assume $a < p$, otherwise the limiting distribution is trivial with all queues being unstable *almost surely*. Packet transmissions occur in time slots with equal duration, and we assume that the Rayleigh fading coefficient is constant during this time. If a packet is successfully received, then the corresponding receiver instantly sends an acknowledgment through an error-free channel and the packet is removed from the queue.

The Signal-to-Interference Ratio[1] at the ith receiver, with $i \in \mathbb{N}$ and associated with receiving a packet at time t, is computed as

$$\mathrm{SIR}_i(t) = \frac{H_{ii}\,\ell(R_i)}{\sum_{j\neq i} e_j \mathbb{1}_{\{Q_j>0\}} H_{ji}\,\ell(||X_j - Y_i||)}, \tag{1}$$

where H_{ij} is exponentially distributed with a parameter 1 and represents the Rayleigh fading coefficient from transmitter j to receiver i, $e_j \in \{0,1\}$ is iid Bernoulli distributed with a parameter p and represents if user j gained access to the channel, and $\mathbb{1}_{\{Q_j>0\}} \in \{0,1\}$ is equal to zero if the queue of user j is empty and one otherwise. Assuming the capture model, the packet is successfully received if $\mathrm{SIR}_i(t)$ is greater than the threshold $\theta \in \mathbb{R}_+$ [6]. Slivnyak's Theorem [4, Th. 1.4.5.] allows us to concentrate on the typical user, which we denote by the index $i = 0$, without changing the distribution of the PPP.

Spatiotemporal correlation leads to the coupling of queues, which results in a highly complicated analysis. Thus, we assume that the interference on the typical receiver is iid across time and use the mean-field approximation [7] to maintain the analytical tractability. This method considers a typical queue and substitutes the interaction with other queues for an effective mean interaction, reducing the problem to the analysis of a single entity and its interdependence with the distribution of the population of entities in each state. For more details about the method, please refer to [1, Page 28].

Let us define the stationary packet success probability of the typical user as

$$p_s(r) \triangleq \lim_{t\to\infty} \mathbb{P}(\mathrm{SIR}_0(t) > \theta). \tag{2}$$

In view of the mean-field approximation, we define the stationary mean queue load $\bar{\rho}$ as the limiting probability of finding the typical transmitter with a nonempty queue, i.e.,

$$\bar{\rho} \triangleq \lim_{t\to\infty} \mathbb{P}(Q_0(t) > 0), \tag{3}$$

and the stationary queue load of a typical transmitter with the link distance equal to r as

$$\rho(r) \triangleq \lim_{t\to\infty} \mathbb{P}(Q_0(t) > 0 \mid R_0 = r), \tag{4}$$

where $Q_0(t) \in \mathbb{N}$ corresponds to the number of packets in the typical queue at time t.

[1]We assume an interference-limited network because, for the most part of large-scale networks, the effect of noise is negligible with respect to that of aggregate interference [1], [5].

III. Stationary Analysis

At first, let us suppose that the limiting distribution exists and the mean queue load converges to $\bar{\rho} \in [0,1]$. Then, at stationary conditions, the active transmitters form a thinned Poisson point process $\Phi_a \subset \Phi$ of a density $\lambda_a = \lambda\, p\, \bar{\rho}$. It is known that the stationary packet success probability of a typical link with a transmission distance $r > 0$ is given by [6]

$$p_s(r) = \exp\left(-J_\ell(r)\,\lambda_a\right), \tag{5}$$

where
$$J_\ell(r) \triangleq \int_{\mathbb{R}_+} \frac{2\pi\, x\, \theta\ell(x)}{\ell(r) + \theta\ell(x)}\,\mathrm{d}x,$$

which is a strictly monotone increasing function and tends to infinity in the positive infinity, because ℓ is a strictly monotone decreasing function and tends to zero in the positive infinity.

Then, from Queueing Theory, the stationary queue load of a typical user with the link distance $r > 0$ is

$$\rho(r) = \min\left\{\frac{a}{p\,p_s(r)}, 1\right\}, \tag{6}$$

because the service rate is given by $p\,p_s(r)$. However, to calculate $p_s(r)$ in (5) we need the density of active users λ_a, which depends on the mean queue load $\bar{\rho}$. This, in turn, depends on $\rho(r)$. The following proposition provides a solution to this problem.

Proposition 1. *If the stationary mean queue load $\bar{\rho} \in [0,1]$ exists, it must satisfy the fixed point equation $T(\bar{\rho}) = \bar{\rho}$, where*

$$T(x) \triangleq 1 - \frac{a}{p}\int_0^{\ln(p/a)} \mathrm{e}^u\, F_*(u/px)\,\mathrm{d}u, \quad x \in (0,1], \tag{7}$$

$T(0) = a/p$, and F_* is the CDF of the random variable $\lambda J_\ell(R)$.

Proof. See Appendix A. □

Because J_ℓ is strictly monotone, it is invertible, and then

$$F_*(x) = \begin{cases} F_R\left(J_\ell^{-1}\left(\frac{x}{\lambda}\right)\right), & \text{if } x > \lambda \lim_{r\to 0^+} J_\ell(r), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

An important remark is that we are dealing with the general path loss function ℓ and a general distribution of link distances R, so care must be taken with F_*, in the sense that the derivative F_*' may not exist.

The following theorems provide results on the existence and uniqueness of a valid stationary mean queue load $\bar{\rho}$.

Theorem 1. *The function T has a fixed point.*

Proof. Because T is an integral of a function, we can show that it is continuous[2]. Further, $T(0) = a/p < 1$ and $a/p \leq T(1) \leq 1$. Thus, the result follows from the Intermediate Value Theorem [8, Th. 4.23.] on $T(x) - x$, $x \in [0,1]$. □

Theorem 2. *If $a > p\,\Omega$, where Ω is the solution of $\Omega\,\mathrm{e}^{\Omega} = 1$, then the function T has a unique fixed point.*

Proof. See Appendix B. □

[2]T is c-Lipschitz continuous. From Appendix B, $c = (p/a)\ln(p/a)$.

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Theorem 3. *If the derivative F'_* exists and $a > p/\mathrm{e}$, then the function T has a unique fixed point.*

Proof. See Appendix C. □

Because $\Omega > 1/\mathrm{e}$, Theorems 2 and 3 complement each other. While one has a larger set of valid functions, the other has a larger set of arrival rates that guarantee uniqueness.

Remarkably, we do not need any information about the distribution of the link distances, the path loss model, the threshold for successful communication, or the density of users to guarantee uniqueness through Theorems 2 and 3 (except for F_* being differentiable in Theorem 3).

Let ε represent the proportion of unstable queues[3] in the network, then in view of the mean-field approximation and Loynes Theorem [10] we can define

$$\varepsilon \triangleq \lim_{t\to\infty} \mathbb{P}\big(p\,\mathbb{P}(\mathrm{SIR}_0(t) < \theta \mid R_0) < a\big). \tag{9}$$

Then, using (5), (8), and Proposition 1, we can show that

$$\varepsilon = 1 - F_*\left(\frac{\ln(p/a)}{p\,\bar{\rho}}\right) \leq \bar{\rho}. \tag{10}$$

This inequality is expected because ε takes into account only the queues with a load equal to 1.

Proposition 2. *If the derivative F'_* exists and $a > p/\mathrm{e}$, then*

$$\frac{\partial\bar{\rho}}{\partial p} < 0 \quad \textit{and} \quad \frac{\partial\varepsilon}{\partial p} < 0.$$

Proof. See Appendix D. □

From the above proposition, if $a \geq 1/\mathrm{e}$, then the access probability p that minimizes the proportion of unstable nodes and the load on the queues is $p = 1$. Other works [2], [11] also reached an equivalent result in a different framework. This appears to be a powerful theoretical result because it is valid for almost any set of system parameters, general path loss model, and general distribution of link distances. However, we must remember that one of the model assumptions is PPP independence of the location of active transmitters across time slots, and this assumption becomes less valid as we increase the value of p [12]. Thus, a more sensible conclusion is that, instead of setting $p = 1$, we should increase p as long as the system model assumptions remain valid. Section IV presents more material on this topic.

Now, let us turn our attention to the scenarios that have multiple limiting distributions, i.e., when T admits more than one fixed point. We can build a sequence that converges to the least or the greatest solution of the fixed point equation, depending on the initial conditions. Let the recurrence equation that defines the sequence $\{\bar{\rho}_n\}_{n\in\mathbb{N}}$ be $\bar{\rho}_{n+1} = T(\bar{\rho}_n), n \in \mathbb{N}$.

Proposition 3. *When $\bar{\rho}_0 = 0$, the sequence $\{\bar{\rho}_n\}_{n\in\mathbb{N}}$ converges to the least solution of $T(\bar{\rho}) = \bar{\rho}$. Otherwise, if $\bar{\rho}_0 = 1$, it converges to the greatest solution.*

Proof. See Appendix E. □

[3]In the literature [9], it is common to define the ε-stability region. However, in the present work, it is more convenient to simply define ε.

Note that the fixed point is unique if and only if the greatest solution is equal to the least solution. This is a simple form to verify whether the stationary distribution is unique when neither of the uniqueness theorems (2 and 3) are satisfied.

Using the concept of dominant networks [2], [9], [13], we can build a physical interpretation of Proposition 3. Suppose a dominant network where all nodes transmit *dummy* packets. The density of active transmitters is $\lambda_a = \lambda p$, and thus, we can calculate the typical queue distributions to find the mean queue load, which we shall denote by $\bar{\rho}_1$. This value can be used as an upper bound to the density of active transmitters in the original network. Then, we can define a new dominant network, whose density of active transmitters is given by $\lambda_a = \lambda p \bar{\rho}_1$ and find the corresponding $\bar{\rho}_2$, and so on. The process described above is equivalent to calculating $\bar{\rho}_{n+1} = T(\bar{\rho}_n)$ with $\bar{\rho}_0 = 1$. We can also perform an analogous reasoning for the lower bound, i.e., we assume that the density of active users is $\lambda_a = \lambda a$, which corresponds to a system operating with a packet success probability equal to 1, and calculate the typical mean queue load in this network, which again we shall denote by $\bar{\rho}_1$. This can be used to estimate another lower bound on the density of active users in the original network, and the process continues analogously to the upper bound case with the sole difference that $\bar{\rho}_0 = a/p$.

IV. Example of a non-unique solution

Here we present a scenario where the conditions of the uniqueness theorems are not satisfied and there is more than one solution to the fixed point equation $T(\bar{\rho}) = \bar{\rho}$.

As commonly chosen in the literature [2], [11], [12], [14], [15], let the path loss function $\ell(r) = r^{-\alpha}$, where $\alpha > 2$, the distribution of R be Rayleigh with mean μ_R, and let us define the auxiliary parameters $\kappa \triangleq \frac{\sin(\delta\pi)}{4\lambda\theta^\delta\delta\pi\mu_R^2}$, $\delta \triangleq 2/\alpha$. Then, $F_*(x) = 1 - \mathrm{e}^{-\kappa x}$ and for $x > 0$,

$$T(x) = \begin{cases} \frac{\left(\frac{a}{p}\right)\frac{\kappa}{px} - \left(\frac{a}{p}\right)^{\frac{\kappa}{px}}}{\frac{\kappa}{px} - 1}, & \text{if } x \neq \kappa/p, \\ \frac{a}{p}\left(1 + \ln\left(\frac{p}{a}\right)\right), & \text{otherwise.} \end{cases} \tag{11}$$

For the simulations we set $a = 0.002$ packets per time slot per node, $p = 0.1$, $\lambda = 1$ node per unit of area, $\alpha = 5$, $\theta = 2$, $\mu_R = 3.457$ units of length, which gives a $\kappa \approx 0.012$. However, any combination of parameters that results in $\kappa < 0.248$ would work, because in this case we can find a and p that give non-unicity of the fixed point, as long as $\alpha > 2$. Nevertheless, when α approaches 2 the simulations become expensive, because we need to increase the number of nodes to properly simulate the large-scale network effect. The simulations are of static Poisson networks and do not consider the simplifying assumptions of the system model, such as mean-field approximation or independence of the SIR across time and space.

We show in Fig. 1 the *attractors* and *repellers*[4] of the chosen scenario, which can be intuitively seen as "stable" and

[4]We define *attractor* as a fixed point x_a of T that satisfies $T^n(x) \to x_a$ as $n \to \infty$ for all x in some neighborhood V_a of x_a, where T^n denotes repeated composition of the function with itself. This neighborhood V_a is called *basin of attraction*. A fixed point x_r of T is a *repeller* if there exist $n_0 \in \mathbb{N}$ and a neighborhood V_r of x_r such that $T^n(x) \notin V_r$ for all $n > n_0, x \in V_r \backslash \{x_r\}$.

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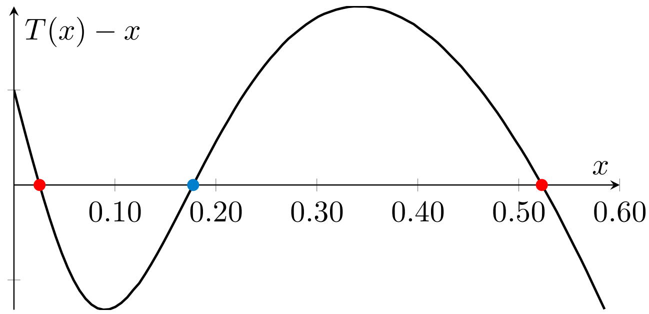

Figure 1. Red dots are the *attractors* and the blue point is the *repeller*.

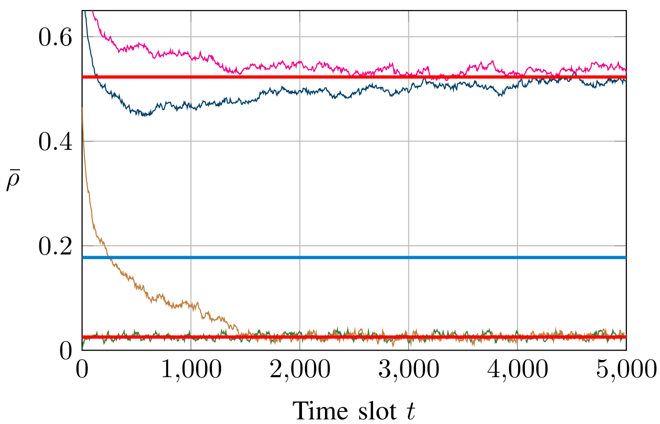

Figure 2. System simulation with four different initial conditions for the queues. We show the average proportion of nonempty queues as an estimation of the value of $\bar{\rho}$. The horizontal red lines are the *attractors* and the blue line is the *repeller*.

"unstable" equilibrium points, respectively. Fig. 2 shows some simulations, which converge for both *attractors* depending on the initial conditions of the queues. For example, we notice that when queues start heavy loaded, the probability to converge to the upper fixed point increases. The brown curve illustrates that even when we start with initial conditions above the *repeller*, the system may still surpass it. This is due to the fact that the initial conditions do not depend only on the initial mean queue load of the system but also on the number of packets on each one of the queues. In addition, there is "noise" in the curves, because we are using a finite number of queues in the simulation, and it tends to vanish as the number of queues tends to infinity.

It is clear that the difference in performance from the upper fixed point to the lower fixed point is immense, e.g., the proportion of unstable queues in one case is $\varepsilon \approx 0.407$, and in the other it is $\varepsilon \approx 9 \cdot 10^{-9}$. Thus, when multiple limiting stationary distributions exist, it is then of key importance to operate in the smallest fixed point.

For that, a simple working strategy is to keep p low, for example $p < \mathrm{e}\, a$ (which satisfies the uniqueness Theorem 3), until reaching a stationary state, then slowly increase p for all nodes. This would improve the performance (Proposition 2) and decrease the probability of converging to another, presumably worse, stationary state. In Fig. 3 we show the theoretical *attractors* and simulate the system using different initial conditions for the queues. Even when employing the strategy of the slowly increasing p, we could not make the system converge to the lower fixed point for $p > 0.4$. That was expected at some point, because when p increases, the *basin of attraction* of the lower fixed point becomes smaller and the stochastic nature of the process makes it exit this region and converge to the upper fixed point.

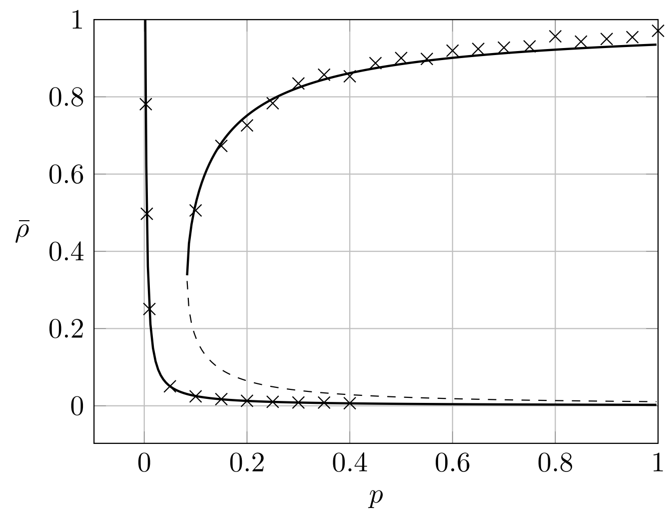

Figure 3. Solution of $T(\bar{\rho}) = \bar{\rho}$ as a function of p. The marks are simulation results. The dashed curve is the *repeller* and divides the *basins of attraction*.

V. Conclusion

Considering a general path loss model and general distribution of link distances, we provided simple sufficient conditions for the uniqueness of a stationary limiting distribution.

The developed mathematical formulation guided us in finding a counterexample to a general uniqueness assumption. We verified through extensive simulations that the proposed scenario indeed possesses two very different limiting stationary distributions, which remain valid even when considering a static Poisson network without the main simplifying assumptions of the original system model.

Furthermore, from the uniqueness theorems, we are inclined to conclude (in a more general setting) that this phenomenon of non-unicity is more likely in light traffic scenarios. So those are the cases one must be most wary.

Appendix A
Proof of Proposition 1

Using (5), (6), integration by parts, and the substitution $u = p\bar{\rho}w$, we have

$$\begin{aligned}\bar{\rho} &= \mathbb{E}\left[\rho(R_0)\right] = \mathbb{E}\left[\min\left\{\frac{a}{p}\mathrm{e}^{\lambda J_\ell(R_0)p\bar{\rho}}, 1\right\}\right] \\ &= 1 - F_*\left(\frac{\ln(p/a)}{p\bar{\rho}}\right) + \frac{a}{p}\int_0^{\ln(p/a)/p\bar{\rho}} \mathrm{e}^{p\bar{\rho}w}\,\mathrm{d}F_*(w) \\ &= 1 - \frac{a}{p}\int_0^{\ln(p/a)} \mathrm{e}^{u}\, F_*(u/p\bar{\rho})\,\mathrm{d}u = T(\bar{\rho}),\end{aligned}$$

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where $\mathrm{d}F(w)$ is the Lebesgue–Stieltjes notation, and if the probability density function exists we can write it as $F'(w)\mathrm{d}w$.

Further, $T(x) \to a/p$ as $x \to 0^+$, because $F_*(y) \to 1$ as $y \to \infty$, which comes from R being a proper random variable.

Appendix B
Proof of Theorem 2

Let us prove for all $x, y \in [\frac{a}{p}, 1]$, $|T(x) - T(y)| < |x - y|$. Then, by the Banach fixed point Theorem [8, Th. 9.23.], the function T admits a unique fixed point.

Without loss of generality let $x < y$, then

$$\begin{aligned}|T(x) - T(y)| &= \frac{a}{p}\left(\int_0^{\ln(p/a)} \mathrm{e}^u F_*(u/px)\mathrm{d}u - \int_0^{\ln(p/a)} \mathrm{e}^u F_*(u/py)\mathrm{d}u\right)\\ &\overset{(i)}{=} a\left(\int_{\ln(p/a)/py}^{\ln(p/a)/px} x\mathrm{e}^{pxw} F_*(w)\mathrm{d}w - \int_0^{\ln(p/a)/py} (y\mathrm{e}^{pyw} - x\mathrm{e}^{pxw}) F_*(w)\mathrm{d}w\right)\\ &\le a\int_{\ln(p/a)/py}^{\ln(p/a)/px} x\mathrm{e}^{pxw}\mathrm{d}w = 1 - \mathrm{e}^{-\ln(p/a)\frac{y-x}{y}} \le \frac{\ln(p/a)}{y}(y - x)\\ &\le \frac{\ln(p/a)}{a/p}|x - y| < \frac{\ln(1/\Omega)}{\Omega}|x - y| = |x - y|.\end{aligned}$$

In (i) we perform the change of variables $u = (px)w$ for the left integral and $u = (py)w$ for the right one, and split the intervals of integration of the left integral.

Appendix C
Proof of Theorem 3

Let $\bar{\rho}$ be a solution to $T(\bar{\rho}) = \bar{\rho}$. Then,

$$\begin{aligned}T'(\bar{\rho}) &= \frac{a}{(p\bar{\rho})^2}\int_0^{\ln(p/a)} u\,\mathrm{e}^u\, F_*'(u/p\bar{\rho})\,\mathrm{d}u\\ &= \frac{1}{\bar{\rho}}\Bigg(\ln(p/a) F_*(\ln(p/a)/p\bar{\rho})\\ &\quad - \frac{a}{p}\int_0^{\ln(p/a)} (1 + u)\mathrm{e}^u F_*(u/p\bar{\rho})\,\mathrm{d}u\Bigg)\\ &\le \frac{\ln(p/a) - (1 - T(\bar{\rho}))}{\bar{\rho}} < \frac{T(\bar{\rho})}{\bar{\rho}} = 1,\end{aligned}$$

where we used integration by parts in the second equality. As T is a continuous function and $(T(x) - x)'|_{x=\bar{\rho}} < 0$, the fixed point equation cannot have more than one root, because $T(x) - x$ always crosses the x-axis downwards.

Appendix D
Proof of Proposition 2

Taking the derivative with respect to p on both sides of the fixed point equation $T(\bar{\rho}) = \bar{\rho}$ and after some tedious manipulations we have

$$\frac{p}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial p} = -\frac{(1 - \ln(p/a))F_*(\Upsilon) + A}{1 - \ln(p/a)F_*(\Upsilon) + A}, \tag{12}$$

where $A \triangleq \frac{a}{p}\int_0^{\ln(p/a)} u\,\mathrm{e}^u F_*(u/p\bar{\rho})\,\mathrm{d}u \ge 0$ and $\Upsilon \triangleq \frac{\ln(p/a)}{p\bar{\rho}}$.

Since $a > p/\mathrm{e}$, then $\ln(p/a) < 1$. Using this, it is easy to see that both the numerator and denominator of (12) are positive. Thus, $\frac{\partial\bar{\rho}}{\partial p} < 0$.

Note that ε monotonically decreases with Υ from (10). Then, let us calculate the derivative of Υ with (12). After some tedious manipulations,

$$\frac{p}{\Upsilon}\frac{\partial\Upsilon}{\partial p} = \frac{1}{\ln(p/a)}\,\frac{1 - \ln(p/a) + A}{1 - \ln(p/a)F_*(\Upsilon) + A}.$$

Again, it is easy to see that both the numerator and the denominator are positive. Thus, $\frac{\partial\Upsilon}{\partial p} > 0$ and $\frac{\partial\varepsilon}{\partial p} < 0$.

Appendix E
Proof of Proposition 3

The function T is monotonic increasing because F_* is a monotonic increasing function. Also, $0 \le T(0)$, then

$$T(0) \le T(T(0)) \le \cdots \le T^n(0) \triangleq T^{n-1}(T(0)).$$

Hence, when $\bar{\rho}_0 = 0$, the sequence $\{\bar{\rho}_n\}_n$ is increasing. Clearly it is also limited, and thus, it converges [8, Th. 3.14.].

Let $x = T(x)$ be a fixed point. We have that $0 \le x$, then $T(0) \le T(x) = x$. Repeating n times we get $T^n(0) \le x$. As $n \to \infty$ we know $T^n(0) = \bar{\rho}_n$ converges, thus it converges to a value smaller or equal to x. But x is an arbitrary fixed point. This concludes the proof for $\bar{\rho}_0 = 0$.

The proof when $\bar{\rho}_0 = 1$ is analogous.

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